Discrete Mathematics

Richard Johnsonbaugh

Eighth Edition

Pearson

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Dedication

To Pat, my wife, for her continuous support through my many book projects, for formally and informally copy-editing my books, for maintaining good cheer throughout, and for preventing all *egregious* mistakes that would have otherwise found their way into print. Her contributions are deeply appreciated.

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Preface

This updated edition is intended for a one- or two-term introductory course in discrete mathematics, based on my experience in teaching this course over many years and requests from users of previous editions. Formal mathematics prerequisites are minimal; calculus is not required. There are no computer science prerequisites. The book includes examples, exercises, figures, tables, sections on problem-solving, sections containing problem-solving tips, section reviews, notes, chapter reviews, self-tests, and computer exercises to help the reader master introductory discrete mathematics. In addition, an Instructor's Guide and website are available.

In the early 1980s there were few textbooks appropriate for an introductory course in discrete mathematics. However, there was a need for a course that extended students' mathematical maturity and ability to deal with abstraction, which also included useful topics such as combinatorics, algorithms, and graphs. The original edition of this book (1984) addressed this need and significantly influenced the development of discrete mathematics courses. Subsequently, discrete mathematics courses were endorsed by many groups for several different audiences, including mathematics and computer science majors. A panel of the Mathematical Association of America (MAA) endorsed a year-long course in discrete mathematics. The Educational Activities Board of the Institute of Electrical and Electronics Engineers (IEEE) recommended a freshman discrete mathematics course. The Association for Computing Machinery (ACM) and IEEE accreditation guidelines mandated a discrete mathematics course. This edition, like its predecessors, includes topics such as algorithms, combinatorics, sets, functions, and mathematical induction endorsed by these groups. It also addresses understanding and constructing proofs and, generally, expanding mathematical maturity.

New to This Edition

The changes in this book, the eighth edition, result from comments and requests from numerous users and reviewers of previous editions of the book. This edition includes the following changes from the seventh edition:

- The web icons in the seventh edition have been replaced by short URLs, making it possible to quickly access the appropriate web page, for example, by using a hand-held device.
- The exercises in the chapter self-tests no longer identify the relevant sections making the self-test more like a real exam. (The hints to these exercises *do* identify the relevant sections.)

- Examples that are worked problems clearly identify where the solution begins and ends.
- The number of exercises in the first three chapters (Sets and Logic; Proofs; and Functions, Sequences, and Relations) has been increased from approximately 1640 worked examples and exercises in the seventh edition to over 1750 in the current edition.
- Many comments have been added to clarify potentially tricky concepts (e.g., "subset" and "element of," collection of sets, logical equivalence of a sequence of propositions, logarithmic scale on a graph).
- There are more examples illustrating diverse approaches to developing proofs and alternative ways to prove a particular result [see, e.g., Examples 2.2.4 and 2.2.8; Examples 6.1.3(c) and 6.1.12; Examples 6.7.7, 6.7.8, and 6.7.9; Examples 6.8.1 and 6.8.2].
- A number of definitions have been revised to allow them to be more directly applied in proofs [see, e.g., one-to-one function (Definition 3.1.22) and onto function (Definition 3.1.29)].
- Additional real-world examples (see descriptions in the following section) are included.
- The altered definition of sequence (Definition 3.2.1) provides more generality and makes subsequent discussion smoother (e.g., the discussion of subsequences).
- Exercises have been added (Exercises 40–49, Section 5.1) to give an example of an algebraic system in which prime factorization does not hold.
- An application of the binomial theorem is used to prove Fermat's little theorem (Exercises 40 and 41, Section 6.7).
- There is now a randomized algorithm to search for a Hamiltonian cycle in a graph (Algorithm 8.3.10).
- The Closest-Pair Problem (Section 13.1 in the seventh edition) has been integrated into Chapter 7 (Recurrence Relations) in the current edition. The algorithm to solve the closest-pair problem is based on merge sort, which is discussed and analyzed in Chapter 7. Chapter 13 in the seventh edition, which has now been removed, had only one additional section.
- A number of recent books and articles have been added to the list of references, and several book references have been updated to current editions.
- The number of exercises has been increased to nearly 4500. (There were approximately 4200 in the seventh edition.)

Contents and Structure

Content Overview

Chapter 1 Sets and Logic
Coverage includes quantifiers and features practical examples such as using the Google search engine (Example 1.2.13). We cover translating between English and symbolic expressions as well as logic in programming languages. We also include a logic game (Example 1.6.15), which offers an alternative way to determine whether a quantified propositional function is true or false.

Chapter 2 Proofs
Proof techniques discussed include direct proofs, counterexamples, proof by contradiction, proof by contrapositive, proof by cases, proofs of equivalence, existence proofs (constructive and nonconstructive), and mathematical induction. We present loop invariants as a practical application of mathematical induction. We also include a brief, optional section on resolution proofs (a proof technique that can be automated).

Chapter 3 Functions, Sequences, and Relations
The chapter includes strings, sum and product notations, and motivating examples such as the Luhn algorithm for computing credit card check digits, which opens the chapter. Other examples include an introduction to hash functions (Example 3.1.15), pseudorandom number generators (Example 3.1.16). a real-world example of function composition showing its use in making a price comparison (Example 3.1.45), an application of partial orders to task scheduling (Section 3.3), and relational databases (Section 3.6).

Chapter 4 Algorithms
The chapter features a thorough discussion of algorithms, recursive algorithms, and the analysis of algorithms. We present a number of examples of algorithms before getting into big-oh and related notations (Sections 4.1 and 4.2), thus providing a gentle introduction and motivating the formalism that follows. We then continue with a full discussion of the "big oh," omega, and theta notations for the growth of functions (Section 4.3). Having all of these notations available makes it possible to make precise statements about the growth of functions and the time and space required by algorithms.

We use the algorithmic approach throughout the remainder of the book. We mention that many modern algorithms do not have all the properties of classical algorithms (e.g., many modern algorithms are not general, deterministic, or even finite). To illustrate the point, we give an example of a randomized algorithm (Example 4.2.4). Algorithms are written in a flexible form of pseudocode, which resembles currently popular languages such as C, C++, and Java. (The book does not assume any computer science prerequisites; the description of the pseudocode used is given in Appendix C.) Among the algorithms presented are:

- Tiling (Section 4.4)
- Euclidean algorithm for finding the greatest common divisor (Section 5.3)
- RSA public-key encryption algorithm (Section 5.4)
- Generating combinations and permutations (Section 6.4)
- Merge sort (Section 7.3)
- Finding a closest pair of points (Section 7.4)
- Dijkstra's shortest-path algorithm (Section 8.4)
- Backtracking algorithms (Section 9.3)
- Breadth-first and depth-first search (Section 9.3)
- Tree traversals (Section 9.6)
- Evaluating a game tree (Section 9.9)
- Finding a maximal flow in a network (Section 10.2)

Chapter 5 Introduction to Number Theory
The chapter includes classical results (e.g., divisibility, the infinitude of primes, fundamental theorem of arithmetic), as well as algorithmic number theory (e.g., the Euclidean algorithm to find the greatest common divisor, exponentiation using repeated squaring, computing s and t such that $\gcd(a, b) = sa + tb$, computing an inverse modulo an inte-

ger). The major application is the RSA public-key cryptosystem (Section 5.4). The calculations required by the RSA public-key cryptosystem are performed using the algorithms previously developed in the chapter.

Chapter 6 Counting Methods and the Pigeonhole Principle
Coverage includes combinations, permutations, discrete probability (optional Sections 6.5 and 6.6), and the Pigeonhole Principle. Applications include internet addressing (Section 6.1) and real-world pattern recognition problems in telemarketing (Example 6.6.21) and virus detection (Example 6.6.22) using Bayes' Theorem.

Chapter 7 Recurrence Relations
The chapter includes recurrence relations and their use in the analysis of algorithms.

Chapter 8 Graph Theory
Coverage includes graph models of parallel computers, the knight's tour, Hamiltonian cycles, graph isomorphisms, and planar graphs. Theorem 8.4.3 gives a simple, short, elegant proof of the correctness of Dijkstra's algorithm.

Chapter 9 Trees
Coverage includes binary trees, tree traversals, minimal spanning trees, decision trees, the minimum time for sorting, and tree isomorphisms.

Chapter 10 Network Models
Coverage includes the maximal flow algorithm and matching.

Chapter 11 Boolean Algebras and Combinatorial Circuits
Coverage emphasizes the relation of Boolean algebras to combinatorial circuits.

Chapter 12 Automata, Grammars, and Languages
Our approach emphasizes modeling and applications. We discuss the *SR* flip-flop circuit in Example 12.1.11, and we describe fractals, including the von Koch snowflake, which can be described by special kinds of grammars (Example 12.3.19).

Book frontmatter and endmatter
Appendixes include coverage of matrices, basic algebra, and pseudocode. A reference section provides more than 160 references to additional sources of information. Front and back endpapers summarize the mathematical and algorithm notation used in the book.

Features of Content Coverage

- **A strong emphasis on the interplay among the various topics.** Examples of this include:
  - We closely tie mathematical induction to recursive algorithms (Section 4.4).
  - We use the Fibonacci sequence in the analysis of the Euclidean algorithm (Section 5.3).
  - Many exercises throughout the book require mathematical induction.
  - We show how to characterize the components of a graph by defining an equivalence relation on the set of vertices (see the discussion following Example 8.2.13).
  - We count the number of nonisomorphic n-vertex binary trees (Theorem 9.8.12).
- **A strong emphasis on reading and doing proofs.** We illustrate most proofs of theorems with annotated figures and/or motivate them by special Discussion sec-

tions. Separate sections (Problem-Solving Corners) show students how to attack and solve problems and how to do proofs. Special end-of-section Problem-Solving Tips highlight the main problem-solving techniques of the section.

- **A large number of applications, especially applications to computer science.**
- **Figures and tables** illustrate concepts, show how algorithms work, elucidate proofs, and motivate the material. Several figures illustrate proofs of theorems. The captions of these figures provide additional explanation and insight into the proofs.

Textbook Structure

Each chapter is organized as follows:

Chapter X Overview
Section X.1
Section X.1 Review Exercises
Section X.1 Exercises
Section X.2
Section X.2 Review Exercises
Section X.2 Exercises
⋮
Chapter X Notes
Chapter X Review
Chapter X Self-Test
Chapter X Computer Exercises

In addition, most chapters have **Problem-Solving Corners** (see “Hallmark Features” for more information about this feature).

Section review exercises review the key concepts, definitions, theorems, techniques, and so on of the section. All section review exercises have answers in the back of the book. Although intended for reviews of the sections, section review exercises can also be used for placement and pretesting.

Chapter notes contain suggestions for further reading. **Chapter reviews** provide reference lists of the key concepts of the chapters. **Chapter self-tests** contain exercises based on material from throughout the chapter, with answers in the back of the book.

Computer exercises include projects, implementation of some of the algorithms, and other programming related activities. Although there is no programming prerequisite for this book and no programming is introduced in the book, these exercises are provided for those readers who want to explore discrete mathematics concepts with a computer.

Hallmark Features

Exercises

The book contains nearly 4500 exercises, approximately 150 of which are computer exercises. We use a star to label exercises felt to be more challenging than average. Exercise numbers in color (approximately one-third of the exercises) indicate that the exercise has a hint or solution in the back of the book. The solutions to most of the remaining exercises may be found in the Instructor’s Guide. A handful of exercises are clearly identified as requiring calculus. No calculus concepts are used in the main body of the book and, except for these marked exercises, no calculus is needed to solve the exercises.

Examples

The book contains almost 650 worked examples. These examples show students how to tackle problems in discrete mathematics, demonstrate applications of the theory, clarify proofs, and help motivate the material.

Problem-Solving Corners

The Problem-Solving Corner sections help students attack and solve problems and show them how to do proofs. Written in an informal style, each is a self-contained section centered around a problem. The intent of these sections is to go beyond simply presenting a proof or a solution to the problem: we show alternative ways of attacking a problem, discuss what to look for in trying to obtain a solution to a problem, and present problem-solving and proof techniques.

Each Problem-Solving Corner begins with a statement of a problem. We then discuss ways to attack the problem, followed by techniques for finding a solution. After we present a solution, we show how to correctly write it up in a formal manner. Finally, we summarize the problem-solving techniques used in the section. Some sections include a Comments subsection, which discusses connections with other topics in mathematics and computer science, provides motivation for the problem, and lists references for further reading about the problem. Some Problem-Solving Corners conclude with a few exercises.

Supplements and Technology

Instructor's Solution Manual (downloadable)

ISBN-10: 0–321-98309-2 | ISBN-13: 978-0–321-98309-1
The Instructor's Guide, written by the author, provides worked-out solutions for most exercises in the text. It is available for download to qualified instructors from the Pearson Instructor Resource Center www.pearsonhighered.com/irc.

Web Support

The short URLs in the margin of the text provide students with direct access to relevant content at point-of-use, including:

- Expanded explanations of difficult material and links to other sites for additional information about discrete mathematics topics.
- Computer programs (in C or C++).

The URL `goo.gl/f03Crh` provides access to all of the above resources plus an errata list for the text.

NOTE:
When you enter URLs that appear in the text, take care to distinguish the following characters:
`l` = lowercase l
`I` = uppercase I
`1` = one
`O` = uppercase O
`0` = zero

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Richard Johnsonbaugh

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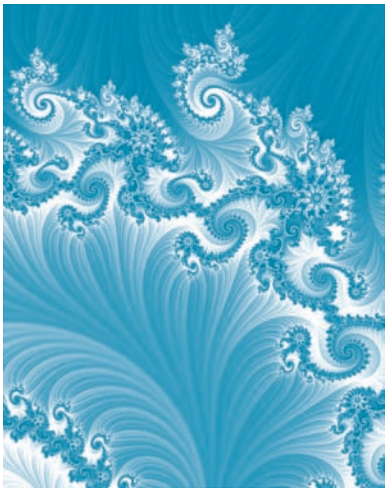

Chapter 1

SETS AND LOGIC

Go Online
For more on logic, see `goo.gl/F7b35e`

Chapter 1 begins with sets. A **set** is a collection of objects; order is not taken into account. Discrete mathematics is concerned with objects such as graphs (sets of vertices and edges) and Boolean algebras (sets with certain operations defined on them). In this chapter, we introduce set terminology and notation. In Chapter 2, we treat sets more formally after discussing proof and proof techniques. However, in Section 1.1, we provide a taste of the logic and proofs to come in the remainder of Chapter 1 and in Chapter 2.

Logic is the study of reasoning; it is specifically concerned with whether reasoning is correct. Logic focuses on the relationship among statements as opposed to the content of any particular statement. Consider, for example, the following argument:

All mathematicians wear sandals.
Anyone who wears sandals is an algebraist.
Therefore, all mathematicians are algebraists.

Technically, logic is of no help in determining whether any of these statements is true; however, if the first two statements are true, logic assures us that the statement,

All mathematicians are algebraists,

is also true.

Logic is essential in reading and developing proofs, which we explore in detail in Chapter 2. An understanding of logic can also be useful in clarifying ordinary writing. For example, at one time, the following ordinance was in effect in Naperville, Illinois: “It shall be unlawful for any person to keep more than three dogs and three cats upon his property within the city.” Was one of the citizens, who owned five dogs and no cats, in violation of the ordinance? Think about this question now; then analyze it (see Exercise 75, Section 1.2) after reading Section 1.2.

1.1 Sets

Go Online
For more on sets, see goo.gl/F7b35e

The concept of set is basic to all of mathematics and mathematical applications. A **set** is simply a collection of objects. The objects are sometimes referred to as elements or members. If a set is finite and not too large, we can describe it by listing the elements in it. For example, the equation

$$A = \{1, 2, 3, 4\} \tag{1.1.1}$$

describes a set A made up of the four elements 1, 2, 3, and 4. A set is determined by its elements and not by any particular order in which the elements might be listed. Thus the set A might just as well be specified as $A = \{1, 3, 4, 2\}$. The elements making up a set are assumed to be distinct, and although for some reason we may have duplicates in our list, only one occurrence of each element is in the set. For this reason we may also describe the set A defined in (1.1.1) as $A = \{1, 2, 2, 3, 4\}$.

If a set is a large finite set or an infinite set, we can describe it by listing a property necessary for membership. For example, the equation

$$B = \{x \mid x \text{ is a positive, even integer}\} \tag{1.1.2}$$

describes the set B made up of all positive, even integers; that is, B consists of the integers 2, 4, 6, and so on. The vertical bar "|" is read "such that." Equation (1.1.2) would be read "B equals the set of all x such that x is a positive, even integer." Here the property necessary for membership is "is a positive, even integer." Note that the property appears after the vertical bar. The notation in (1.1.2) is called **set-builder notation**.

A set may contain *any* kind of elements whatsoever, and they need *not* be of the same "type." For example,

$$\{4.5, \text{Lady Gaga}, \pi, 14\}$$

is a perfectly fine set. It consists of four elements: the number 4.5, the person Lady Gaga, the number $\pi (= 3.1415\ldots)$, and the number 14.

A set may contain elements that are themselves sets. For example, the set

$$\{3, \{5, 1\}, 12, \{\pi, 4.5, 40, 16\}, \text{Henry Cavill}\}$$

consists of five elements: the number 3, the set $\{5, 1\}$, the number 12, the set $\{\pi, 4.5, 40, 16\}$, and the person Henry Cavill.

Some sets of numbers that occur frequently in mathematics generally, and in discrete mathematics in particular, are shown in Figure 1.1.1. The symbol **Z** comes from the German word, *Zahlen*, for *integer*. Rational numbers are quotients of integers, thus **Q** for *quotient*. The set of real numbers **R** can be depicted as consisting of all points on a straight line extending indefinitely in either direction (see Figure 1.1.2).[†]

Symbol	*Set*	*Example of Members*
Z	Integers	$-3, 0, 2, 145$
Q	Rational numbers	$-1/3, 0, 24/15$
R	Real numbers	$-3, -1.766, 0, 4/15, \sqrt{2}, 2.666\ldots, \pi$

Figure 1.1.1 Sets of numbers.

[†]The real numbers can be constructed by starting with a more primitive notion such as "set" or "integer," or they can be obtained by stating properties (axioms) they are assumed to obey. For our purposes, it suffices to think of the real numbers as points on a straight line. The construction of the real numbers and the axioms for the real numbers are beyond the scope of this book.

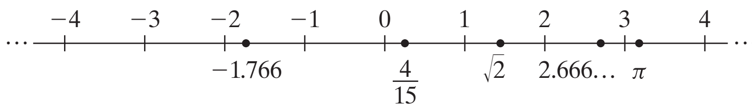

Figure 1.1.2 The real number line.

To denote the negative numbers that belong to one of **Z**, **Q**, or **R**, we use the superscript minus. For example, $\mathbf{Z}^-$ denotes the set of negative integers, namely $-1, -2, -3, \ldots$. Similarly, to denote the positive numbers that belong to one of the three sets, we use the superscript plus. For example, $\mathbf{Q}^+$ denotes the set of positive rational numbers. To denote the nonnegative numbers that belong to one of the three sets, we use the superscript *nonneg*. For example, $\mathbf{Z}^{nonneg}$ denotes the set of nonnegative integers, namely $0, 1, 2, 3, \ldots$.

If X is a finite set, we let $|X| =$ number of elements in X. We call $|X|$ the **cardinality** of X. There is also a notion of cardinality of infinite sets, although we will not discuss it in this book. For example, the cardinality of the integers, Z, is denoted $\aleph_0$, read "aleph null." Aleph is the first letter of the Hebrew alphabet.

Example 1.1.1 For the set A in (1.1.1), we have $|A| = 4$, and the cardinality of A is 4. The cardinality of the set $\{\mathbf{R}, \mathbf{Z}\}$ is 2 since it contains two elements, namely the two sets **R** and **Z**. ◀

Given a description of a set X such as (1.1.1) or (1.1.2) and an element x, we can determine whether or not x belongs to X. If the members of X are listed as in (1.1.1), we simply look to see whether or not x appears in the listing. In a description such as (1.1.2), we check to see whether the element x has the property listed. If x is in the set X, we write $x \in X$, and if x is not in X, we write $x \notin X$. For example, $3 \in \{1, 2, 3, 4\}$, but $3 \notin \{x \mid x \text{ is a positive, even integer}\}$.

The set with no elements is called the **empty** (or **null** or **void**) **set** and is denoted $\varnothing$. Thus $\varnothing = \{\ \}$.

Two sets X and Y are **equal** and we write $X = Y$ if X and Y have the same elements. To put it another way, $X = Y$ if the following two conditions hold:

- For every x, if $x \in X$, then $x \in Y$,

and

- For every x, if $x \in Y$, then $x \in X$.

The first condition ensures that every element of X is an element of Y, and the second condition ensures that every element of Y is an element of X.

Example 1.1.2 If $A = \{1, 3, 2\}$ and $B = \{2, 3, 2, 1\}$, by inspection, A and B have the same elements. Therefore $A = B$. ◀

Example 1.1.3 Show that if $A = \{x \mid x^2 + x - 6 = 0\}$ and $B = \{2, -3\}$, then $A = B$.

SOLUTION According to the criteria in the paragraph immediately preceding Example 1.1.2, we must show that for every x,

$$\text{if } x \in A, \text{ then } x \in B, \tag{1.1.3}$$

and for every x,

$$\text{if } x \in B, \text{ then } x \in A. \tag{1.1.4}$$

To verify condition (1.1.3), suppose that $x \in A$. Then

$$x^2 + x - 6 = 0.$$

Solving for x, we find that $x = 2$ or $x = -3$. In either case, $x \in B$. Therefore, condition (1.1.3) holds.

To verify condition (1.1.4), suppose that $x \in B$. Then $x = 2$ or $x = -3$. If $x = 2$, then

$$x^2 + x - 6 = 2^2 + 2 - 6 = 0.$$

Therefore, $x \in A$. If $x = -3$, then

$$x^2 + x - 6 = (-3)^2 + (-3) - 6 = 0.$$

Again, $x \in A$. Therefore, condition (1.1.4) holds. We conclude that $A = B$. ◀

For a set X to *not* be equal to a set Y (written $X \neq Y$), X and Y must *not* have the same elements: There must be at least one element in X that is not in Y or at least one element in Y that is not in X (or both).

Example 1.1.4 Let $A = \{1, 2, 3\}$ and $B = \{2, 4\}$. Then $A \neq B$ since there is at least one element in A (1 for example) that is not in B. [Another way to see that $A \neq B$ is to note that there is at least one element in B (namely 4) that is not in A.] ◀

Suppose that X and Y are sets. If every element of X is an element of Y, we say that X is a **subset** of Y and write $X \subseteq Y$. In other words, X is a subset of Y if for every x, if $x \in X$, then $x \in Y$.

Example 1.1.5 If $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$, by inspection, every element of C is an element of A. Therefore, C is a subset of A and we write $C \subseteq A$. ◀

Example 1.1.6 Let $X = \{x \mid x^2 + x - 2 = 0\}$. Show that $X \subseteq \mathbf{Z}$.

SOLUTION We must show that for every x, if $x \in X$, then $x \in \mathbf{Z}$. If $x \in X$, then $x^2 + x - 2 = 0$. Solving for x, we obtain $x = 1$ or $x = -2$. In either case, $x \in \mathbf{Z}$. Therefore, for every x, if $x \in X$, then $x \in \mathbf{Z}$. We conclude that X is a subset of $\mathbf{Z}$ and we write $X \subseteq \mathbf{Z}$. ◀

Example 1.1.7 The set of integers $\mathbf{Z}$ is a subset of the set of rational numbers $\mathbf{Q}$. If $n \in \mathbf{Z}$, n can be expressed as a quotient of integers, for example, $n = n/1$. Therefore $n \in \mathbf{Q}$ and $\mathbf{Z} \subseteq \mathbf{Q}$. ◀

Example 1.1.8 The set of rational numbers $\mathbf{Q}$ is a subset of the set of real numbers $\mathbf{R}$. If $x \in \mathbf{Q}$, x corresponds to a point on the number line (see Figure 1.1.2) so $x \in \mathbf{R}$. ◀

For X to *not* be a subset of Y, there must be at least one member of X that is not in Y.

Example 1.1.9 Let $X = \{x \mid 3x^2 - x - 2 = 0\}$. Show that X is not a subset of $\mathbf{Z}$.

SOLUTION If $x \in X$, then $3x^2 - x - 2 = 0$. Solving for x, we obtain $x = 1$ or $x = -2/3$. Taking $x = -2/3$, we have $x \in X$ but $x \notin \mathbf{Z}$. Therefore, X is not a subset of $\mathbf{Z}$. ◀

Any set X is a subset of itself, since any element in X is in X. Also, the empty set is a subset of every set. If $\varnothing$ is *not* a subset of some set Y, according to the discussion preceding Example 1.1.9, there would have to be at least one member of $\varnothing$ that is not in Y. But this cannot happen because the empty set, by definition, has no members.

Notice the difference between the terms "subset" and "element of." The set X is a *subset* of the set $Y (X \subseteq Y)$, if every element of X is an element of Y; x is an *element of* $X (x \in X)$, if x is a member of the set X.

Example 1.1.10 Let $X = \{1, 3, 5, 7\}$ and $Y = \{1, 2, 3, 4, 5, 6, 7\}$. Then $X \subseteq Y$ since every element of X is an element of Y. But $X \notin Y$, since the *set* X is not a member of Y. Also, $1 \in X$, but 1 is not a subset of X. Notice the difference between the number 1 and the *set* $\{1\}$. The set $\{1\}$ *is* a subset of X. ◀

If X is a subset of Y and X does not equal Y, we say that X is a **proper subset** of Y and write $X \subset Y$.

Example 1.1.11 Let $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$. Then C is a proper subset of A since C is a subset of A but C does not equal A. We write $C \subset A$. ◀

Example 1.1.12 Example 1.1.7 showed that $\mathbf{Z}$ is a subset of $\mathbf{Q}$. In fact, $\mathbf{Z}$ is a proper subset of $\mathbf{Q}$ because, for example, $1/2 \in \mathbf{Q}$, but $1/2 \notin \mathbf{Z}$. ◀

Example 1.1.13 Example 1.1.8 showed that $\mathbf{Q}$ is a subset of $\mathbf{R}$. In fact, $\mathbf{Q}$ is a proper subset of $\mathbf{R}$ because, for example, $\sqrt{2} \in \mathbf{R}$, but $\sqrt{2} \notin \mathbf{Q}$. (In Example 2.2.3, we will show that $\sqrt{2}$ is not the quotient of integers). ◀

The set of all subsets (proper or not) of a set X, denoted $\mathcal{P}(X)$, is called the **power set** of X.

Example 1.1.14 If $A = \{a, b, c\}$, the members of $\mathcal{P}(A)$ are

$$\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

All but $\{a, b, c\}$ are proper subsets of A. ◀

In Example 1.1.14, $|A| = 3$ and $|\mathcal{P}(A)| = 2^3 = 8$. In Section 2.4 (Theorem 2.4.6), we will give a formal proof that this result holds in general; that is, the power set of a set with n elements has 2^n elements.

Given two sets X and Y, there are various set operations involving X and Y that can produce a new set. The set

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

is called the **union** of X and Y. The union consists of all elements belonging to either X or Y (or both).

The set

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$

is called the **intersection** of X and Y. The intersection consists of all elements belonging to both X and Y.

The set

$$X - Y = \{x \mid x \in X \text{ and } x \notin Y\}$$

is called the **difference** (or **relative complement**). The difference $X - Y$ consists of all elements in X that are not in Y.

Example 1.1.15 If $A = \{1, 3, 5\}$ and $B = \{4, 5, 6\}$, then

$$\begin{aligned} A \cup B &= \{1, 3, 4, 5, 6\} \\ A \cap B &= \{5\} \\ A - B &= \{1, 3\} \\ B - A &= \{4, 6\}. \end{aligned}$$

Notice that, in general, $A - B \neq B - A$. ◀

Example 1.1.16 Since $\mathbf{Q} \subseteq \mathbf{R}$,

$$\begin{aligned} \mathbf{R} \cup \mathbf{Q} &= \mathbf{R} \\ \mathbf{R} \cap \mathbf{Q} &= \mathbf{Q} \\ \mathbf{Q} - \mathbf{R} &= \varnothing. \end{aligned}$$

The set $\mathbf{R} - \mathbf{Q}$, called the set of **irrational numbers,** consists of all real numbers that are not rational. ◀

We call a set $\mathcal{S}$, whose elements are sets, a **collection of sets** or a **family of sets**. For example, if

$$\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\},$$

then $\mathcal{S}$ is a collection or family of sets. The set $\mathcal{S}$ consists of the sets

$$\{1, 2\}, \{1, 3\}, \{1, 7, 10\}.$$

Sets X and Y are **disjoint** if $X \cap Y = \varnothing$. A collection of sets $\mathcal{S}$ is said to be **pairwise disjoint** if, whenever X and Y are distinct sets in $\mathcal{S}$, X and Y are disjoint.

Example 1.1.17 The sets $\{1, 4, 5\}$ and $\{2, 6\}$ are disjoint. The collection of sets $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$ is pairwise disjoint. ◀

Sometimes we are dealing with sets, all of which are subsets of a set U. This set U is called a **universal set** or a **universe.** The set U must be explicitly given or inferred from the context. Given a universal set U and a subset X of U, the set $U - X$ is called the **complement** of X and is written $\overline{X}$.

Example 1.1.18 Let $A = \{1, 3, 5\}$. If U, a universal set, is specified as $U = \{1, 2, 3, 4, 5\}$, then $\overline{A} = \{2, 4\}$. If, on the other hand, a universal set is specified as $U = \{1, 3, 5, 7, 9\}$, then $\overline{A} = \{7, 9\}$. The complement obviously depends on the universe in which we are working. ◀

Example 1.1.19 Let the universal set be $\mathbf{Z}$. Then $\overline{\mathbf{Z}^-}$, the complement of the set of negative integers, is $\mathbf{Z}^{nonneg}$, the set of nonnegative integers. ◀

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Venn diagrams provide pictorial views of sets. In a Venn diagram, a rectangle depicts a universal set (see Figure 1.1.3). Subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set. In Figure 1.1.3 we see two sets A and B within the universal set U. Region 1 represents $\overline{(A \cup B)}$, the elements in neither A nor B. Region 2 represents $A - B$, the elements in A but not in B. Region 3 represents $A \cap B$, the elements in both A and B. Region 4 represents $B - A$, the elements in B but not in A.

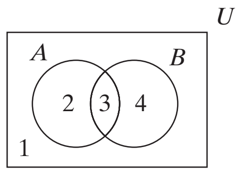

Figure 1.1.3 A Venn diagram.

Example 1.1.20 Particular regions in Venn diagrams are depicted by shading. The set $A \cup B$ is shown in Figure 1.1.4, and Figure 1.1.5 represents the set $A - B$. ◀

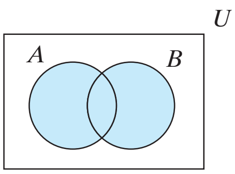

Figure 1.1.4 A Venn diagram of $A \cup B$.

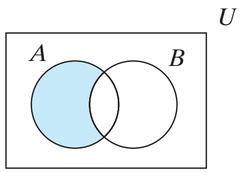

Figure 1.1.5 A Venn diagram of $A - B$.

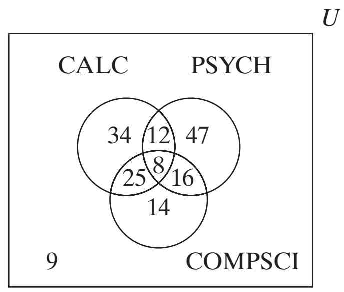

Figure 1.1.6 A Venn diagram of three sets CALC, PSYCH, and COMPSCI. The numbers show how many students belong to the particular region depicted.

To represent three sets, we use three overlapping circles (see Figure 1.1.6).

Example 1.1.21 Among a group of 165 students, 8 are taking calculus, psychology, and computer science; 33 are taking calculus and computer science; 20 are taking calculus and psychology; 24 are taking psychology and computer science; 79 are taking calculus; 83 are taking psychology; and 63 are taking computer science. How many are taking none of the three subjects?

SOLUTION Let CALC, PSYCH, and COMPSCI denote the sets of students taking calculus, psychology, and computer science, respectively. Let U denote the set of all 165 students (see Figure 1.1.6). Since 8 students are taking calculus, psychology, and computer science, we write 8 in the region representing $\text{CALC} \cap \text{PSYCH} \cap \text{COMPSCI}$. Of the 33 students taking calculus and computer science, 8 are also taking psychology; thus 25 are taking calculus and computer science but not psychology. We write 25 in the region representing $\text{CALC} \cap \overline{\text{PSYCH}} \cap \text{COMPSCI}$. Similarly, we write 12 in the region representing $\text{CALC} \cap \text{PSYCH} \cap \overline{\text{COMPSCI}}$ and 16 in the region representing $\overline{\text{CALC}} \cap \text{PSYCH} \cap \text{COMPSCI}$. Of the 79 students taking calculus, 45 have now been accounted for. This leaves 34 students taking only calculus. We write 34 in the region representing $\text{CALC} \cap \overline{\text{PSYCH}} \cap \overline{\text{COMPSCI}}$. Similarly, we write 47 in the region representing $\overline{\text{CALC}} \cap \text{PSYCH} \cap \overline{\text{COMPSCI}}$ and 14 in the region representing

$\overline{\text{CALC}} \cap \overline{\text{PSYCH}} \cap \text{COMPSCI}$. At this point, 156 students have been accounted for. This leaves 9 students taking none of the three subjects. ◀

Venn diagrams can also be used to visualize certain properties of sets. For example, by sketching both $\overline{(A \cup B)}$ and $\overline{A} \cap \overline{B}$ (see Figure 1.1.7), we see that these sets are equal. A formal proof would show that for every x, if $x \in \overline{(A \cup B)}$, then $x \in \overline{A} \cap \overline{B}$, and if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{(A \cup B)}$. We state many useful properties of sets as Theorem 1.1.22.

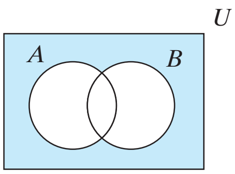

Figure 1.1.7 The shaded region depicts both $\overline{(A \cup B)}$ and $\overline{A} \cap \overline{B}$; thus these sets are equal.

Theorem 1.1.22

Let U be a universal set and let A, B, and C be subsets of U. The following properties hold.

(a) Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$$

(b) Commutative laws:

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

(c) Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(d) Identity laws:

$$A \cup \varnothing = A, \quad A \cap U = A$$

(e) Complement laws:

$$A \cup \overline{A} = U, \quad A \cap \overline{A} = \varnothing$$

(f) Idempotent laws:

$$A \cup A = A, \quad A \cap A = A$$

(g) Bound laws:

$$A \cup U = U, \quad A \cap \varnothing = \varnothing$$

(h) Absorption laws:

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

(i) Involution law:

$$\overline{\overline{A}} = A^{\dagger}$$

(j) 0/1 laws:

$$\overline{\varnothing} = U, \quad \overline{U} = \varnothing$$

(k) De Morgan's laws for sets:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B}, \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

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For a biography of De Morgan, see `goo.gl/F7b35e`

Proof The proofs are left as exercises (Exercises 46–56, Section 2.1) to be done after more discussion of logic and proof techniques. ◀

We define the union of a collection of sets $\mathcal{S}$ to be those elements x belonging to at least one set X in $\mathcal{S}$. Formally,

$$\cup\mathcal{S} = \{x \mid x \in X \text{ for some } X \in \mathcal{S}\}.$$

† $\overline{\overline{A}}$ denotes the complement of the complement of A, that is, $\overline{\overline{A}} = \overline{(\overline{A})}$.

Similarly, we define the intersection of a collection of sets $\mathcal{S}$ to be those elements x belonging to every set X in $\mathcal{S}$. Formally,

$$\cap \mathcal{S} = \{x \mid x \in X \text{ for all } X \in \mathcal{S}\}.$$

Example 1.1.23 Let $\mathcal{S} = \{\{1, 2\}, \{1, 3\}, \{1, 7, 10\}\}$. Then $\cup\mathcal{S} = \{1, 2, 3, 7, 10\}$ since each of the elements 1, 2, 3, 7, 10 belongs to at least one set in $\mathcal{S}$, and no other element belongs to any of the sets in $\mathcal{S}$. Also $\cap\mathcal{S} = \{1\}$ since only the element 1 belong to every set in $\mathcal{S}$. ◀

If

$$\mathcal{S} = \{A_1, A_2, \ldots, A_n\},$$

we write

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{n} A_i, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{n} A_i,$$

and if

$$\mathcal{S} = \{A_1, A_2, \ldots\},$$

we write

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i.$$

Example 1.1.24 For $i \geq 1$, define $A_i = \{i, i+1, \ldots\}$ and $\mathcal{S} = \{A_1, A_2, \ldots\}$. As examples, $A_1 = \{1, 2, 3, \ldots\}$ and $A_2 = \{2, 3, 4, \ldots\}$. Then

$$\bigcup \mathcal{S} = \bigcup_{i=1}^{\infty} A_i = \{1, 2, \ldots\}, \qquad \bigcap \mathcal{S} = \bigcap_{i=1}^{\infty} A_i = \varnothing.$$ ◀

A partition of a set X divides X into nonoverlapping subsets. More formally, a collection $\mathcal{S}$ of nonempty subsets of X is said to be a **partition** of the set X if every element in X belongs to exactly one member of $\mathcal{S}$. Notice that if $\mathcal{S}$ is a partition of X, $\mathcal{S}$ is pairwise disjoint and $\cup\mathcal{S} = X$.

Example 1.1.25 Since each element of $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ is in exactly one member of $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}, \{7, 8\}\}$, $\mathcal{S}$ is a partition of X. ◀

At the beginning of this section, we pointed out that a set is an unordered collection of elements; that is, a set is determined by its elements and not by any particular order in which the elements are listed. Sometimes, however, we do want to take order into account. An **ordered pair** of elements, written (a, b), is considered distinct from the ordered pair (b, a), unless, of course, $a = b$. To put it another way, $(a, b) = (c, d)$ precisely when $a = c$ and $b = d$. If X and Y are sets, we let $X \times Y$ denote the set of all ordered pairs (x, y) where $x \in X$ and $y \in Y$. We call $X \times Y$ the **Cartesian product** of X and Y.

Example 1.1.26 If $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then

$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$

$$X \times X = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

$$Y \times Y = \{(a, a), (a, b), (b, a), (b, b)\}.$$ ◀